Notes on Matrix Valued Paraproducts

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Abstract Denote by M_n the algebra of $n\times n$ matrices. We consider the dyadic paraproducts π_b associated with M_n valued functions b, and show that the $L^\infty(M_n)$ norm of b does not dominate $||\pi_b||_{L^2(\ell^2_n)\to L^2(\ell^2_n)}$ uniformly over n. We also consider paraproducts associated with noncommutative martingales and prove that their boundedness on bounded noncommutative L^p-martingale spaces implies their boundedness on bounded noncommutative L^q-martingale spaces for all $1<p<q<\infty$.

1 Introduction

Denote by M_n the algebra of $n\times n$ matrices. Let $(\mathbb{T},\mathcal{F}_k,dt)$ be the unit circle with Haar measure and the usual dyadic filtration. Let b be an M_n valued function on $\mathbb{T}$. The matrix valued dyadic paraproduct associated with b, denoted by π_b, is the operator defined as

$$\pi_b(f)=\sum_k (d_kb)(E_{k-1}f), \quad \forall f\in L^2(\ell^2_n). \tag{1.1}$$

Here E_kf is the conditional expectation of f with respect to $\mathcal{F}_k$, i.e. the unique $\mathcal{F}_k$-measurable function such that

$$\int_F E_kf dt=\int_F f dt, \quad \forall F\in\mathcal{F}_k.$$

And d_kb is defined to be $E_kb-E_{k-1}b$.

In the classical case (when b is a scalar valued function), paraproducts are usually considered as dyadic singular integrals and play important roles in the proof of the classical T(1) theorem. It is well known that

$$\|\pi_b\|_{L^2\to L^2}\backsimeq\|b\|_{BMO_d},$$

where BMO$_d$ denotes the dyadic BMO norm defined as

$$\|b\|_{BMO_d}=\sup_m\|E_m\sum_{k=m}^{\infty}|d_kb|^2||^{\frac12}_{L^\infty}.$$

And by the Calderón-Zygmund decomposition and the Marcinkiewicz interpolation theorem, we have $||\pi_b||_{L^p\to L^p}\backsimeq||\pi_b||_{L^p\to L^p}\backsimeq||b||_{BMO_d}$ for all $1<p<\infty$.

When b is M_n valued, it was proved by Katz ([4]) and independently by Nazarov, Treil and Volberg ([8], see [10] for another proof by Pisier) that

$$\|\pi_b\|_{L^2(\ell^2_n)\to L^2(\ell^2_n)}\le c\log(n+1)\,\|b\|_{BMO_c}. \tag{1.2}$$

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Here $\|\cdot\|_{\mathrm{BMO}_c}$ is the column BMO norm defined by

$$\|b\|_{BMO_c} = \sup_m \left\| E_m \sum_{k=m}^{\infty} (d_k b)^*(d_k b) \right\|_{L^\infty(M_n)}^{\frac{1}{2}},$$

where $(d_k b)^*$ is the adjoint of $d_k b$. Nazarov, Pisier, Treil and Volberg ([7]) proved later that the constant $c \log(n+1)$ in (1.2) is optimal. Thus the BMO_c norm does not dominate $\|\pi_b\|_{L^2(\ell_n^2)\to L^2(\ell_n^2)}$ uniformly over n.

Can we expect something weaker? In particular, does there exist a constant c independent of n such that, for every $n \in \mathbb{N}$,

$$\|\pi_b\|_{L^2(\ell_n^2)\to L^2(\ell_n^2)} \leq c\,\|b\|_{L^\infty(M_n)}? \tag{1.3}$$

Some known facts made (1.3) look hopeful. For example, the Hankel operator associated with the M_n valued function b has a norm equivalent to $||b||_{(H^1(S^1))^*}$. Here $||\cdot||_{(H^1(S^1))^*}$ denotes the dual norm on the trace class valued Hardy space $H^1(S^1)$. And S. Petermichl proved a close relation between π_b and the Hankel operators associated with b (see [9]).

In this paper, we prove the following theorem, which shows there does not exist any constant c independent of n such that (1.3) holds.

Theorem 1.1 *For every $n \in \mathbb{N}$, there exists an M_n valued function b with $\|b\|_{L^\infty(M_n)} \leq 1$ but such that*

$$\|\pi_b\|_{L^2(\ell_n^2)\to L^2(\ell_n^2)} \geq c\log(n+1),$$

where $c > 0$ is independent of n.

This also gives a new proof that the constant $c\log(n+1)$ in (1.2) is optimal.

Denote by S^p the Schatten p class on ℓ^2. For $f \in L^p(S^p)$, we define $\pi_b(f)$ as in (1.1) also. As pointed out in [10], it is easy to check that $\|\pi_b\|_{L^2(S^2)\to L^2(S^2)} = \|\pi_b\|_{L^2(\ell^2)\to L^2(\ell^2)}$. For scalar valued b, as we mentioned previously, we have $||\pi_b||_{L^p\to L^p} \backsimeq ||\pi_b||_{L^q\to L^q}$. We wonder if this is still true for matrix valued b, i.e. if π_b's boundedness on $L^p(S^p)$ implies their boundedness on $L^q(S^q)$ for all $1 < p, q < \infty$.

More generally, we can consider paraproducts associated with noncommutative martingales. Let $\mathcal{M}$ be a finite von Neumann algebra with a normalized faithful trace τ. For $1 \leq p < \infty$, we denote by $L^p(\mathcal{M})$ the noncommutative L^p space associated with $(\mathcal{M}, \tau)$. Recall the norm in $L^p(\mathcal{M})$ is defined as

$$\|f\|_p = (\tau|x|^p)^{\frac{1}{p}}, \quad \forall f \in L^p(\mathcal{M}),$$

where $|f| = (f^* f)^{\frac{1}{2}}$. For convenience, we usually set $L^\infty(\mathcal{M}) = \mathcal{M}$ equipped with the operator norm $\|\cdot\|_{\mathcal{M}}$. Let $\mathcal{M}_k$ be an increasing filtration of von Neumann subalgebras of $\mathcal{M}$ such that $\cup_{k\geq 0}\mathcal{M}_k$ generates $\mathcal{M}$ in the w$^*-$ topology. Denote by E_k the conditional expectation of $\mathcal{M}$ with respect to $\mathcal{M}_k$. E_k is a norm 1 projection of $L^p(\mathcal{M})$ onto

$L^p(\mathcal{M}_k)$. For $1 \le p \le \infty$, a sequence $f = (f_k)_{k\ge 0}$ with $f_k \in L^p(\mathcal{M}_k)$ is called a bounded noncommutative L^p-martingale, denoted by $(f_k)_{k\ge 0} \in L^p(\mathcal{M})$, if $E_k f_m = f_k, \forall k \le m$ and

$$||(f_k)_{k\ge 0}||_{L^p(\mathcal{M})} = \sup_k ||f_k||_{L^p(\mathcal{M})} < \infty.$$

Because of the uniform convexity of the space $L^p(\mathcal{M})$, for $1 < p < \infty$, we can and will identify the space of all bounded $L^p(\mathcal{M})$-martingales with $L^p(\mathcal{M})$ itself. In particular, for any $f \in L^p(\mathcal{M})$, set $f_k = E_k f$, then $f = (f_k)_{k\ge 0}$ is a bounded $L^p(\mathcal{M})$-martingale and $||(f_k)_{k\ge 0}||_{L^p(\mathcal{M})} = ||f||_{L^p(\mathcal{M})}$. Denote by $d_k f = E_k f - E_{k-1} f$.

We say an increasing filtration $\mathcal{M}_k$ is "regular" if there exists a constant $c > 0$ such that, for any $m, a \in \mathcal{M}_m, a \ge 0$,

$$||a||_\infty \le c||E_{m-1} a||_\infty.$$

For $\mathcal{M}$ with a regular filtration $\mathcal{M}_k$, $b \in L^2(\mathcal{M})$, we define paraproducts $\pi_b, \widetilde{\pi}_b$ as operators for bounded $L^p(\mathcal{M})$ $(1 < p < \infty)$-martingales $f = (f_k)_{k\ge 0}$ as

$$\pi_b(f) = \sum_k d_k b f_{k-1}, \qquad \widetilde{\pi}_b(f) = \sum_k f_{k-1} d_k b.$$

We prove the following result for π_b and $\widetilde{\pi}_b$:

Theorem 1.2 *Let $1 < p < q < \infty$, if $\widetilde{\pi}_b$ and π_b are both bounded on $L^p(\mathcal{M})$ then they are both bounded on $L^q(\mathcal{M})$.*

We still do not know what happens when $p > q$.

2 Proof of Theorem 1.1 and Application to "Sweep" functions.

Denote by tr the usual trace on M_n and $S_n^p (1 \le p < \infty)$ the Schatten p classes on ℓ_n^2.
Proof of Theorem 1.1. Let $c(n)$ be the best constant such that

$$\|\pi_b\|_{L^2(\ell_n^2) \to L^2(\ell_n^2)} \le c(n) \, \|b\|_{L^\infty(M_n)}, \quad \forall b \in L^\infty(M_n).$$

Denote by T the triangle projection on S_n^1, we are going to show

$$\|T\|_{S_n^1 \to S_n^1} \le c(n).$$

Once this is proved, we are done since $\|T\|_{S_n^1 \to S_n^1} \backsim \log(n+1)$ (see [5]). Note that every A in the unit ball of S_n^1 can be written as

$$A = \sum_m \lambda^{(m)} \alpha^{(m)} \otimes \beta^{(m)}$$

with $\sum_m \lambda^{(m)} \le 1$, $\sup_m\{||\alpha^{(m)}||_{\ell^2_n}, ||\beta^{(m)}||_{\ell^2_n}\} \le 1$. Therefore, we only need to show

$$\|T(\alpha \otimes \beta)\|_{S^1_n} \le c(n) \, \|\alpha\|_{\ell^2_n} \, \|\beta\|_{\ell^2_n}\,, \quad \forall \alpha = (\alpha_k)_k, \beta = (\beta_k)_k \in \ell^2_n. \tag{2.4}$$

Let D be the diagonal M_n valued function defined as

$$D = \sum_{i=1}^{n} r_i e_i \otimes e_i$$

where r_i is the i-th Rademacher function on $\mathbb{T}$ and $(e_i)_{i=1}^n$ is the canonical basis of ℓ^2_n. Given $\alpha = (\alpha_k)_k, \beta = (\beta_k)_k \in \ell^2_n$, let

$$f = D\alpha, g = D\beta.$$

Then $f, g \in L^2(\ell^2_n)$, and

$$\|f\|_{L^2(\ell^2_n)} = \|\alpha\|_{\ell^2_n}\,, \|g\|_{L^2(\ell^2_n)} = \|\beta\|_{\ell^2_n}\,. \tag{2.5}$$

It is easy to verify

$$\sum_k E_{k-1} f \otimes d_k g = D(\sum_{i<j\le n} \alpha_i \beta_j e_i \otimes e_j) D.$$

and

$$\left\| \sum_k E_{k-1} f \otimes d_k g \right\|_{L^1(S^1_n)} = \left\| \sum_{i<j\le n} \alpha_i \beta_j e_i \otimes e_j \right\|_{S^1_n} = \|T(\alpha \otimes \beta)\|_{S^1_n}\,. \tag{2.6}$$

On the other hand, by duality between $L^1(S^1_n)$ and $L^\infty(M_n)$, we have,

$$\begin{aligned}
\left\| \sum_k E_{k-1} f \otimes d_k g \right\|_{L^1(S^1_n)} &= \sup\{\ tr \int \sum_k d_k b(E_{k-1} f \otimes d_k g), \|b\|_{L^\infty(M_n)} \le 1\} \\
&\le \sup\{\ \|\pi_b(f)\|_{L^2(\ell^2_n)} \|g\|_{L^2(\ell^2_n)}, \|b\|_{L^\infty(M_n)} \le 1\} \\
&\le c(n) \, \|f\|_{L^2(\ell^2_n)} \, \|g\|_{L^2(\ell^2_n)}\,.
\end{aligned} \tag{2.7}$$

Combining (2.7), (2.5) and (2.6) we get (2.4) and the proof is complete. ∎

Recall that the square function of b is defined as

$$S(b) = (\sum_k |d_k b|^2)^{\frac{1}{2}}.$$

The so called "sweep" function is just the square of the square function, for this reason we denote it by $S^2(b)$,

$$S^2(b) = \sum_k |d_k b|^2.$$

In the classical case, we know that

$$\begin{aligned} ||S(b)||_{BMO_d} &\leq c||b||_{BMO_d} && (2.8)\\ ||S^2(b)||_{BMO_d} &\leq c||b||^2_{BMO_d} && (2.9)\end{aligned}$$

When considering square functions $S(b)$ for M_n valued functions b, a similar result remains true with an absolute constant.

Proposition 2.3 *For any $n \in \mathbb{N}$, and any M_n valued function b, we have*

$$||S(b)||_{BMO_c} \leq \sqrt{2}||b||_{BMO_c}$$

Proof. Since we are in the dyadic case, we have

$$\begin{aligned} ||S(b)||^2_{BMO_c} &\leq 2\sup_m ||E_m[(S(b) - E_mS(b))^*(S(b) - E_mS(b))]||_{L^\infty(M_n)}\\ &= 2\sup_m ||E_mS^2(b) - (E_mS(b))^2||_{L^\infty(M_n)}\end{aligned}$$

Note

$$E_mS^2(b) - \sum_{k=1}^m |d_kb|^2 \geq E_mS^2(b) - (E_mS(b))^2 \geq 0.$$

We get

$$\begin{aligned} ||S(b)||^2_{BMO_c} &\leq 2\sup_m ||E_mS^2(b) - \sum_{k=1}^m |d_kb|^2||_{L^\infty(M_n)}\\ &= 2\sup_m ||E_m \sum_{k=m+1} |d_kb|^2||_{L^\infty(M_n)}\\ &\leq 2||b||^2_{BMO_c}. \blacksquare\end{aligned}$$

Matrix valued sweep functions have been studied in [1], [2] etc. Unlike in the case of square functions, it is proved in [1] that the best constant c_n such that

$$||S^2(b)||_{BMO_c} \leq c_n||b||^2_{BMO_c} \tag{2.10}$$

is $c\log(n+1)$. The following result shows that the best constant c_n is still $c\log(n+1)$ even if we replace $||\cdot||_{BMO_c}$ by the bigger norm $||\cdot||_{L^\infty(M_n)}$ in the right side of (2.10).

Theorem 2.4 *For every $n \in \mathbb{N}$, there exists an M_n valued function b with $\|b\|_{L^\infty(M_n)} \leq 1$ but such that*

$$\left\|S^2(b)\right\|_{BMO_c} \geq c\log(n+1).$$

Proof. Consider a function b that works for the statement of Theorem 1.1. Then $\|b\|_{L^\infty(M_n)} \le 1$ and there exists a function $f \in L^2(S_n^2)$, such that $\|f\|_{L^2(S_n^2)} \le 1$ and

$$\left\| \sum_k d_k b E_{k-1} f \right\|_{L^2(S_n^2)} \ge c \log(n+1). \tag{2.11}$$

We compute the square of the left side of (2.11) and get

$$\begin{aligned}
& \left\| \sum_k d_k b E_{k-1} f \right\|^2_{L^2(S_n^2)} \\
= & \; tr \int \sum_k |d_k b|^2 E_{k-1} f E_{k-1} f^* \\
= & \; tr \int \sum_k |d_k b|^2 (\sum_{i<k} |d_i f^*|^2 + \sum_{i<k} E_{i-1} f d_i f^* + \sum_{i<k} d_i f E_{i-1} f^*) \\
= & \; tr \int \sum_i (\sum_{k>i} |d_k b|^2) |d_i f^*|^2 + tr \int \sum_i (\sum_{k>i} |d_k b|^2)(E_{i-1} f d_i f^* + d_i f E_{i-1} f^*) \\
= & \; I + II
\end{aligned}$$

For I, note $|d_i f^*|^2$ is $\mathcal{F}_i$ measurable, we have

$$\begin{aligned}
I &= tr \int \sum_i E_i (\sum_{k>i} |d_k b|^2) |d_i f^*|^2 \\
&\le \sup_i ||E_i (\sum_{k>i} |d_k b|^2)||_{L^\infty(M_n)} (tr \int \sum_i |d_i f^*|^2) \\
&\le ||b||^2_{BMO_c} ||f||^2_{L^2(S_n^2)} \le 4
\end{aligned}$$

For II, note $E_{i-1} f d_i f^* + d_i f E_{i-1} f^*$ is a martingale difference and $\sum_{k\le i} |d_k|^2$ is $\mathcal{F}_{i-1}$ measurable since we are in the dyadic case, we get

$$\begin{aligned}
II &= tr \int \sum_i S^2(b)(E_{i-1} f d_i f^* + d_i f E_{i-1} f^*) \\
&= tr \int \sum_i d_i(S^2(b))(E_{i-1} f d_i f^* + d_i f E_{i-1} f^*) \\
&\le 2 || \sum_i d_i (S^2(b)) E_{i-1} f ||_{L^2(S_n^2)} ||f||_{L^2(S_n^2)} \\
&\le 2 ||\pi_{S^2(b)}||_{L^2(S_n^2) \to L^2(S_n^2)} \\
&\le 2c \log(n+1) ||S^2(b)||_{BMO_c}.
\end{aligned}$$

We used (1.2) in the last step. Combining this with (2.11), we get

$$c \log(n+1) \le \left\| \sum_k d_k b E_{k-1} f \right\|^2_{L^2(S_n^2)} \le 4 + 2c \log(n+1) ||S^2(b)||_{BMO_c}$$

Thus

$$||S^2(b)||_{\mathrm{BMO}_c} \geq c\log(n+1).$$

This completes the proof. ∎

3 Proof of Theorem 1.2.

We keep the notations introduced in the end of Section 1. Recall BMO spaces of noncommutative martingales are defined for $x = (x_k) \in L^2(\mathcal{M})$ as below (see [?], [?]):

$$\begin{aligned}
\mathrm{BMO}_c(\mathcal{M}) &= \{x : ||x||_{\mathrm{BMO}_c(\mathcal{M})} = \sup_n \left\| E_n |\sum_{k=n}^{\infty} d_k x|^2 \right\|_{\mathcal{M}}^{\frac{1}{2}} < \infty\};\\
\mathrm{BMO}_r(\mathcal{M}) &= \{x : ||x||_{\mathrm{BMO}_r(\mathcal{M})} = ||x^*||_{\mathrm{BMO}_c(\mathcal{M})} < \infty\};\\
\mathrm{BMO}_{cr}(\mathcal{M}) &= \{x : ||x||_{\mathrm{BMO}_{cr}(\mathcal{M})} = \max\{||x||_{\mathrm{BMO}_c(\mathcal{M})}, ||x||_{\mathrm{BMO}_r(\mathcal{M})}\} < \infty\}.
\end{aligned}$$

When $\mathcal{M} = L^\infty(M_n)$, $\mathrm{BMO}_c(\mathcal{M})$ is just BMO_c considered in Section 1 and 2. In this section, for noncommutative martingale b, we consider π_b and $\widetilde{\pi}_b$ as operators on bounded noncommutative L^p-martingale spaces introduced in Section 1. We will need the following interpolation result and the John-Nirenberg theorem for noncommutative martingales proved by Junge and Musat recently (see [3], [6]).

Theorem 3.5 *(Musat) For $1 \leq p \leq q < \infty$,*

$$(BMO_{cr}(\mathcal{M}), L_p(\mathcal{M}))_\theta = L_q(\mathcal{M}), \ \textit{with}\ \theta = \frac{p}{q}.$$

Theorem 3.6 *(Junge, Musat) For any $1 \leq q < \infty$ and any $g = (g_k)_k \in BMO_{cr}(\mathcal{M})$, there exist $c_q, c'_q > 0$ such that*

$$c'_q ||g||_{BMO_{cr}} \leq \sup_{m\in\mathbb{N}} \sup_{a\in\mathcal{M}_m, \tau(|a|^q)\leq 1} \{||\sum_{k\geq m} d_k g a||_{L^q(\mathcal{M})}, ||\sum_{k\geq m} a d_k g||_{L^q(\mathcal{M})}\} \leq c_q ||g||_{BMO_{cr}}. \tag{3.12}$$

In fact, the formula above is proved for $q \geq 2$ in [3]. It is not hard to show that it is also true for $1 \leq q < 2$. In the following, we give a simpler proof of it in the tracial case.

Proof. Note for any $g \in BMO_{cr}(\mathcal{M})$,

$$||g||_{BMO_{cr}(\mathcal{M})} = \sup_{m\in\mathbb{N}} \sup_{a\in\mathcal{M}_m, \tau(|a|^2)\leq 1} \{||\sum_{k\geq m} d_k g a||_{L^2(\mathcal{M})}, ||\sum_{k\geq m} a d_k g||_{L^2(\mathcal{M})}\}.$$

We get $c_2 = c'_2 = 1$. Note for p, r, s with $1/p = 1/r + 1/s$ and $a \in L^p(\mathcal{M}), ||a||_{L^p(\mathcal{M})} \leq 1$, there exist b, c such that $a = bc$ and $||b||_{L^r(\mathcal{M})} \leq 1, ||c||_{L^s(\mathcal{M})} \leq 1$. By Hölder's inequality we then get $c_q = 1$ for $1 \leq q < 2$ and $c'_q = 1$ for $2 < q < \infty$. Thus for $2 < q < \infty$, we

only need to prove the second inequality of (3.12). And, for $1 \le q < 2$, we only need to prove the first inequality of (3.12). Fix $g \in BMO_{cr}(\mathcal{M})$, $m \in \mathbb{N}$, consider the left multiplier L_m and the right multiplier R_m defined as

$$L_m(a) = \sum_{k \ge m} d_k g a \text{ and } R_m(a) = \sum_{k \ge m} a d_k g, \quad \forall a \in \mathcal{M}_m.$$

It is easy to check that

$$\begin{aligned}
\sup_m ||L_m||_{L^2(\mathcal{M}_m) \to L^2(\mathcal{M})} &= ||g||_{BMO_c}, \\
\sup_m ||L_m||_{L^\infty(\mathcal{M}_m) \to BMO_{cr}} &\le ||g||_{BMO_{cr}}; \\
\sup_m ||R_m||_{L^2(\mathcal{M}_m) \to L^2(\mathcal{M})} &= ||g||_{BMO_r}, \\
\sup_m ||R_m||_{L^\infty(\mathcal{M}_m) \to BMO_{cr}} &\le ||g||_{BMO_{cr}}.
\end{aligned}$$

Thus L_m, R_m extend to bounded operators from $L^2(\mathcal{M}_m)$ to $L^2(\mathcal{M})$, as well as from $L^\infty(\mathcal{M}_m)$ to $BMO_{cr}(\mathcal{M})$. By Musat's interpolation result Theorem 3.5, we get L_m and R_m are bounded from $L^q(\mathcal{M}_m)$ to $L^q(\mathcal{M})$ and their operator norms are smaller than $c_q||g||_{BMO_{cr}}$, for all $2 \le q < \infty$. By taking supremum over m, we prove the second inequality of (3.12) for $q \ge 2$.

For $1 \le q < 2$, by interpolation again, for $\theta = \frac{q}{2}$ and some $c''_q > 0$,

$$\begin{aligned}
||L_m||_{L^2(\mathcal{M}_m) \to L^2(\mathcal{M})} &\le c''_q ||L_m||^{\theta}_{L^q(\mathcal{M}_m) \to L^q(\mathcal{M})} ||L_m||^{1-\theta}_{L^\infty(\mathcal{M}_m) \to BMO_{cr}} \\
&\le c''_q ||L_m||^{\theta}_{L^q(\mathcal{M}_m) \to L^q(\mathcal{M})} ||g||^{1-\theta}_{BMO_{cr}}, \\
||R_m||_{L^2(\mathcal{M}_m) \to L^2(\mathcal{M})} &\le c''_q ||R_m||^{\theta}_{L^q(\mathcal{M}_m) \to L^q(\mathcal{M})} ||R_m||^{1-\theta}_{L^\infty(\mathcal{M}_m) \to BMO_{cr}} \\
&\le c''_q ||R_m||^{\theta}_{L^q(\mathcal{M}_m) \to L^q(\mathcal{M})} ||g||^{1-\theta}_{BMO_{cr}}.
\end{aligned}$$

Thus

$$\begin{aligned}
||g||_{BMO_{cr}} &= \max\{\sup_m ||L_m||_{L^2(\mathcal{M}_m) \to L^2(\mathcal{M})}, \sup_m ||R_m||_{L^2(\mathcal{M}_m) \to L^2(\mathcal{M})}\} \\
&\le c''_q ||g||^{1-\theta}_{BMO_{cr}} \sup_m \{||L_m||^{\theta}_{L^q(\mathcal{M}_m) \to L^q(\mathcal{M})}, ||R_m||^{\theta}_{L^q(\mathcal{M}_m) \to L^q(\mathcal{M})}\}.
\end{aligned}$$

This gives the first inequality of (3.12) with $c'_q = (c''_q)^{-\frac{1}{\theta}}$ for $1 \le q < 2$. ∎

Recall that we say a filtration $\mathcal{M}_k$ is "regular" if, for some $c > 0$, $||a||_\infty \le c||E_{m-}a||_\infty$, $\forall m \in \mathbb{N}, a \ge 0, a \in \mathcal{M}_m$.

Lemma 3.7 *For any regular filtration $\mathcal{M}_k$, we have*

$$||b||_{BMO_{cr}(\mathcal{M})} \le c_p \max\{||\pi_b||_{L^p(\mathcal{M}) \to L^p(\mathcal{M})}, ||\widetilde{\pi}_b||_{L^p(\mathcal{M}) \to L^p(\mathcal{M})}\}, \quad \forall 1 \le p < \infty. \tag{3.13}$$

Proof. Note, for any $b \in BMO_{cr}(\mathcal{M})$ with respect to the regular filtration $\mathcal{M}_k$,

$$||b||_{BMO_{cr}(\mathcal{M})} \le c \sup_{m \in \mathbb{N}} \sup_{\tau a^2 \le 1, a \in \mathcal{M}_m} \{||\sum_{k > m} d_k b a||_{L^2(\mathcal{M})}, ||\sum_{k > m} a d_k b||_{L^2(\mathcal{M})}\}.$$

Similar to the proof of Theorem 3.6, we can get,

$$c_q'||b||_{BMO_{cr}} \leq \sup_{m\in\mathbb{N}} \sup_{a\in\mathcal{M}_m,\tau|a|^q\leq 1} \{||\sum_{k>m} d_k ba||_{L^q(\mathcal{M})}, ||\sum_{k>m} ad_k b||_{L^q(\mathcal{M})}\} \leq c_q||b||_{BMO_{cr}}. \tag{3.14}$$

On the other hand, by considering $\pi_b(a), \widetilde{\pi}_b(a)$ for $a \in \mathcal{M}_m$, $||a||_{L^p(\mathcal{M})} \leq 1$, we have

$$\begin{aligned} &\sup_{a\in\mathcal{M}_m,\tau|a|^q\leq 1} \{||\sum_{k>m} d_k ba||_{L^p(\mathcal{M})}, ||\sum_{k>m} ad_k b||_{L^p(\mathcal{M})}\} \\ &\quad\leq 2\max\{||\pi_b||_{L^p(\mathcal{M})\to L^p(\mathcal{M})}, ||\widetilde{\pi}_b||_{L^p(\mathcal{M})\to L^p(\mathcal{M})}\}. \end{aligned}$$

Taking supremum over m in the inequality above, we get (3.13) by (3.14) . ∎

Lemma 3.8 *For $1 < p < \infty$, we have*

$$\|\pi_b\|_{L^\infty(\mathcal{M})\to BMO_{cr}(\mathcal{M})} \leq c_p(\|\pi_b\|_{L^p(\mathcal{M})\to L^p(\mathcal{M})} + ||b||_{BMO_r(\mathcal{M})}). \tag{3.15}$$

$$\|\widetilde{\pi}_b\|_{L^\infty(\mathcal{M})\to BMO_{cr}(\mathcal{M})} \leq c_p(\|\widetilde{\pi}_b\|_{L^p(\mathcal{M})\to L^p(\mathcal{M})} + ||b||_{BMO_c(\mathcal{M})}). \tag{3.16}$$

Proof. We prove (3.15) only. Fix a $f \in L^\infty(\mathcal{M})$ with $\|f\|_{L^\infty(\mathcal{M})} \leq 1$. We have

$$\begin{aligned} &\left\|E_m \sum_{k\geq m} |d_k b E_{k-1} f|^2\right\|_{L^\infty(\mathcal{M})} \\ = &\ \sup\{\tau E_m \sum_{k\geq m} |d_k b E_{k-1} f|^2 a,\ a \in \mathcal{M}_m, a \geq 0, \tau a \leq 1\} \\ = &\ \sup\{\tau \sum_{k\geq m} (d_k b E_{k-1} f a^{\frac{1}{p}})^*(d_k b E_{k-1} f a^{\frac{1}{q}}),\ a \in \mathcal{M}_m, a \geq 0, \tau a \leq 1\} \\ \leq &\ \sup_a \left\| d_m b E_{m-1} f a^{\frac{1}{p}} + \sum_{k>m} d_k b E_{k-1}(f a^{\frac{1}{p}})\right\|_{L^p(\mathcal{M})} \left\|\sum_{k\geq m} d_k b E_{k-1} f a^{\frac{1}{q}}\right\|_{L^q(\mathcal{M})} \end{aligned}$$

Note $||d_m b E_{m-1} f a^{\frac{1}{p}}||_{L^p(\mathcal{M})} \leq ||d_m b||_{\mathcal{M}} \leq ||b||_{BMO_r}$. By (3.12) we get

$$\left\|E_m \sum_{k\geq m} |d_k b E_{k-1} f|^2\right\|_{L^\infty(\mathcal{M})} \leq c_q(||b||_{BMO_r} + \|\pi_b\|_{L^p(\mathcal{M})\to L^p(\mathcal{M})})\,\|\pi_b(f)\|_{BMO_{cr}(\mathcal{M})} \tag{3.17}$$

Taking supremum over m in (3.17), we get

$$\|\pi_b(f)\|^2_{BMO_c(\mathcal{M})} \leq c_q(||b||_{BMO_r} + \|\pi_b\|_{L^p(\mathcal{M})\to L^p(\mathcal{M})})\,\|\pi_b(f)\|_{BMO_{cr}(\mathcal{M})}\,.$$

On the other hand, since $(E_{m-1}f)(E_{m-1}f)^* \leq 1$, we have

$$\|\pi_b(f)\|_{BMO_r(\mathcal{M})} \leq \|b\|_{BMO_r(\mathcal{M})}\,.$$

Thus,

$$\|\pi_b(f)\|^2_{BMO_{cr}(\mathcal{M})} \leq (c_q+1)(\|\pi_b\|_{L^p(\mathcal{M})\to L^p(\mathcal{M})} + ||b||_{BMO_r(\mathcal{M})}) \|\pi_b(f)\|_{BMO_{cr}(\mathcal{M})},$$

Therefore

$$\|\pi_b\|_{L^\infty(\mathcal{M})\to BMO_{cr}(\mathcal{M})} \leq (c_q+1)(\|\pi_b\|_{L^p(\mathcal{M})\to L^p(\mathcal{M})} + ||b||_{BMO_r(\mathcal{M})}). \quad \blacksquare$$

Proof of Theorem 1.2. By Lemma 3.7 and Lemma 3.8 we get immediately that

$$\begin{aligned} &\max \quad \{\|\pi_b\|_{L^\infty(\mathcal{M})\to BMO_{cr}}, \|\widetilde{\pi}_b\|_{L^\infty(\mathcal{M})\to BMO_{cr}}\} \\ \leq c_p &\max \quad \{\|\pi_b\|_{L^p(\mathcal{M})\to L^p(\mathcal{M})}, \quad \|\widetilde{\pi}_b\|_{L^p(\mathcal{M})\to L^p(\mathcal{M})}\} \end{aligned}$$

By the interpolation results of noncommutative martingales(Theorem 3.5), we get

$$\begin{aligned} &\max \quad \{\|\pi_b\|_{L^q(\mathcal{M})\to L^q(\mathcal{M})}, \|\widetilde{\pi}_b\|_{L^q(\mathcal{M})\to L^q(\mathcal{M})}\} \\ \leq c_p &\max \quad \{\|\pi_b\|_{L^p(\mathcal{M})\to L^p(\mathcal{M})}, \|\widetilde{\pi}_b\|_{L^p(\mathcal{M})\to L^p(\mathcal{M})}\}, \end{aligned}$$

for all $1 < p < q < \infty$.

Question : Assume $\pi_b, \widetilde{\pi}_b$ are of type (p,p), are they of weak type $(1,1)$? More precisely, assume $||\pi_b||_{L^p(\mathcal{M})\to L^p(\mathcal{M})} + ||\widetilde{\pi}_b||_{L^p(\mathcal{M})\to L^p(\mathcal{M})} < \infty$, does there exist a constant $C > 0$ such that, for any $f \in L^1(\mathcal{M})$, $\lambda > 0$, there is a projection $e \in \mathcal{M}$ such that

$$\tau(e^\perp) \leq C\frac{||f||_{L^1(\mathcal{M})}}{\lambda} \quad \text{and} \quad ||e\pi_b(f)e||_{L^\infty(\mathcal{M})} + ||e\widetilde{\pi}_b(f)e||_{L^\infty(\mathcal{M})} \leq \lambda?$$

We have the following corollary by applying results of this section to matrix valued dyadic paraproducts discussed in Section 1 and Section 2. Note M_n valued dyadic martingales on the unit circle are noncommutative martingales associated with the von Neuman algebra $\mathcal{M} = L^\infty(\mathbb{T}) \otimes M_n$ and the filtration $\mathcal{M}_k = L^\infty(\mathbb{T}, \mathcal{F}_k) \otimes M_n$.

Corollary 3.9 *Let $1 < p < \infty$, denote by $c_p(n)$ the best constant such that*

$$\|\pi_b\|_{L^p(S_n^p)\to L^p(S_n^p)} \leq c_p(n)\, \|b\|_{L^\infty(M_n)}, \ \forall b.$$

Then

$$c_p(n) \backsim \log(n+1).$$

Proof. Note in the proof of Theorem 1.1, if we see f as a column matrix valued function and g as a row matrix valued function, we will have

$$||f||_{L^p(S_n^p)} = ||\alpha||_{\ell_n^2}, \quad ||g||_{L^q(S_n^q)} = ||\beta||_{\ell_n^2}.$$

By the same method, we can prove $c_p(n) \geq c\log(n+1)$ for all $1 < p < \infty$. For the inverse relation, by (1.2) we have $c_2(n) \leq c\log(n+1)$. Then, by (3.15), we get

$$\begin{aligned} \|\pi_b\|_{L^\infty(M_n)\to BMO_{cr}} &\leq c_2(c_2(n)\,\|b\|_{L^\infty(M_n)} + ||b||_{BMO_{cr}}) \\ &\leq c\log(n+1)||b||_{L^\infty(M_n)}, \quad \forall b \in L^\infty(M_n) \end{aligned} \tag{3.18}$$

Denote by π_b^* the adjoint operator of the dyadic paraproduct π_b, then

$$\pi_b^*(f) = \sum_k (d_k b)^* E_{k-1} f.$$

Note we have the decomposition

$$\pi_b^*(f) = b^* f - \pi_{b^*}(f) - (\pi_{f^*}(b))^*.$$

By (3.18), we get

$$\begin{aligned} \|\pi_b^*\|_{L^\infty(M_n)\to BMO_{cr}} &\leq ||b^*||_{L^\infty(M_n)} + c\log(n+1)||b^*||_{L^\infty(M_n)} + c\log(n+1)||b||_{L^\infty(M_n)} \\ &\leq c\log(n+1)\,\|b\|_{L^\infty(M_n)}\,. \end{aligned} \tag{3.19}$$

By (3.18), (3.19) and the interpolation result Theorem 3.5, we get

$$\|\pi_b\|_{L^p(S_n^p)\to L^p(S_n^p)} \leq c_p \log(n+1)\,\|b\|_{L^\infty(M_n)}\,, \quad \forall 1 < p < \infty.$$

Therefore, we can conclude $c_p(n) \backsim \log(n+1)$. ∎

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